Young Mathematicians Workshop: Calculus of Variations

List of Abstracts

Annika Bach: (Technical University of Eindhoven)

Title: Discrete-to-continuum variational analysis for Ising and Spin models

Abstract: This seminar will be kind of an introductory talk to discrete-to-continuum variational analysis. Here the general objective is to characterise the macroscopic behaviour of energy-driven atomistic systems while keeping the scale-dependent relevant microscopic information. This is done via Γ-convergence, which allows to coarse-grain discrete functionals to their continuum counterparts as the discretisation parameter vanishes. In this talk we will focus on some models with a particularly rich energy landscape, namely Ising and spin models such as the classical XY-model. In the first part I will give a short introduction to the topic and review some of the classical results in this context. In the second part I will present some recent results obtained in collaboration with Marco Cicalese, Leonard Kreutz, and Gianluca Orlando on the antiferromagnetic XY-model on the triangular lattice. Depending on time the last part of the talk will deal with some problems in this context that we would like to solve but we are not yet able to do so.

Marco Bresciani: (FAU Erlangen-Nürnberg)

Title: Variational models with Eulerian-Lagrangian formulation allowing for material failure

Abstract: Variational models featuring mixed Eulerian-Lagrangian formulations arise naturally in many multi- physics problems. The underlying energy functional depends on the deformation of the body, classically set on the reference configuration, and an Eulerian map defined on the deformed configuration in the actual space. The latter has often to comply with nonlinear constraints determined by the context. Concrete examples concern the modeling of liquid crystals and magnetic elastomers, where the Eulerian map represents the nematic director and the magnetization field, respectively.

In this talk, we present an existence theory for minimizers of abstract mixed Eulerian-Lagrangian energies. Building upon previous works by Müller and Spector (1995), and Henao and Mora-Corral (2010-2012), our results move beyond the purely elastic case and investigate models that allow for material failure.

First, we focus on soft materials possibly exhibiting cavitation, that is, the abrupt formation of voids inside the material in response to mechanical stresses. Subsequently, we investigate brittle materials that might experience fracture. Deformations will be modeled as Sobolev maps and special maps with bounded variation, respectively. The regularity of Eulerian maps will be specified in each of these settings according to the geometric and topological properties of the deformed configuration. Effectiveness and limitations of the theory will be illustrated by means of explicit examples covering relevant applications. This talk is based on joint work with Manuel Friedrich (FAU Erlangen-Nürnberg) and Carlos Mora-Corral (Universidad Autónoma de Madrid).

Luca Briani: (Technical University of Munich)

Title: A shape optimization problem involving the torsion function and a related hexagonal structure.

Abstract: For a domain $\Omega \subset \mathbb{R}^d$ with finite measure the torsion function of Ω is defined to be the unique solution $w_{p,\Omega}$ to the following boundary value problem:

$$-\Delta_p w = 1\,, \text{ in } \Omega\,, \quad w = 0 \text{ on } \partial\Omega$$

In *shape optimization* theory $w_{p,\Omega}$ is the simplest example of *state function* and it is important to determine its properties. In this seminar I will discuss the optimization problems for the following *mean-to-max* functional

$$\mathfrak{e}_p(\Omega) = \frac{\int_\Omega w_{p,\Omega}\,\mathrm{d}x}{|\Omega| \|w_{p,\Omega}\|_{L^\infty(\Omega)}}\,, \quad p \in (1, \infty)\,.$$

The latter is known in the literature as the *efficiency* of the torsion function. In the limit case $p = +\infty$ I will show the maximality of an honeycomb-type structure. At last I will describe some applications to the problems of comparing the torsional rigidity $T_p(\Omega)$ with the first eigenvalue of the p-Laplace operator.

References:

[1] Briani, L., Bucur, D. Mean-to-max ratio of the torsion function and honeycomb structures Calc. Var. 62, 198 (2023)

Janusz Ginster: (Humboldt University of Berlin)

Title: Formation of Microstructure for Singularly Perturbed Problems: From Helimagnets to Shape-Memory Alloys and Domain Dependence

Abstract: In this talk we present recent results on scalar-valued variational models for pattern formation in helimagnetic compounds and in shape memory alloys. Precisely, we consider a non-convex multi-well bulk energy term on the unit square, which favors four gradients $(\pm\alpha, \pm\beta)$, regularized by a singular perturbation in terms of the total variation of the second derivative. We derive scaling laws for the minimal energy in the case of an incompatible affine boundary condition in terms of the singular perturbation parameter as well as the ratio α/β and the incompatibility of the boundary condition. We discuss how well-studied models for martensitic microstructures in shape-memory alloys arise as a limiting case, and relations between the different models in terms of scaling laws. Moreover,

we discuss scaling laws for the minimal energy for the specific four gradients $(\pm 1, \pm 1)$ on a rotated square under 0 boundary conditions in terms of the singular perturbation parameter and the angle that measures the distance to the square rotated by 45 degrees, which is fully compatible with the boundary conditions.

This is joint work with B. Zwicknagl.

Rosella Giorgio: (Technical University of Vienna)

Title: Nonlocal justification of antisymmetric exchange functionals.

Abstract: In this talk we consider a nonlocal-to-local analysis in the context of micromagnetic phenomena. We introduce nonlocal antisymmetric interaction energies acting as approximations of the so-called Dzyaloshinskii-Moriya interaction, which in turn constitutes a peculiar term of the micromagnetic energy functional. In a first stage, the nonlocal justification of the antisymmetric functionals is given by a pointwise convergence result, relying on a similar approach to the well-known Bourgain-Brezis-Mironescu formula. Finally, also a Γ-convergence argument is presented. (This is a joint work with E. Davoli and G. Di Fratta).

Leon Happ: (Technical University of Vienna)

Title: A scale independent continuous extension operator for maps on microperforated domains mapping into a target manifold

Abstract: Motivated by examples in micro-magnetics and plasticity, I am currently studying, together with Chiara Gavioli, the existence of continuous extension operators for Sobolev maps on perforated domains. The novelty in our case is that these maps are subject to a constrained in the target space, namely, they are only allowed to range in some prescribed manifold. We investigate whether there exists an extension operator that conserves this property and which is, additionally, independent of the scale of the perforations. Such an extension operator would prove to be of immense benefit in the field of Homogenization, e.g., for deducing compactness results. The application of a special projection operator (see below) calls for a case differentiation in regard to the relation between the integrability p and the dimension n of the domain. Our main result concerns the case that p is smaller than n, and we are able to indeed proof the existence of a continuous extension operator for $W^{1,p}$ maps on perforated domains that preserves the described manifold constrained in the target space, under the assumption that the target manifold is $[p] - 1$ connected. The unconstrained case was already treated in an influential paper by E. Acerbi, V. C. Piat, G. Dal Maso, and D. Percivale from 1992, providing a positive answer. Our work builds upon their result, combining it with an appropriate projection operator firstly suggested by R. Hardt and F.-H. Lin in 1987. The study of this projection operator is related to the field of Obstruction Theory in Homology and is intriguing in its own right. I will discuss both of these ingredients in some detail, and then how they can be combined to obtain the alluded result. Additionally, I will touch upon the connection between this problem and the surjectivity of the trace operator for Sobolev maps between manifolds, as studied for example

very prominently in the already mentioned paper of R. Hardt and F.-H. Lin from 1987, by F. Bethuel and F. Demengel in 1995, and again by F. Bethuel in 2014. Dropping the necessity that the extension operator should be continuous, it turns out that both questions are closely entangled and even equivalent in some cases, depending again on the relation between p and n.

Tim Heilmann: (Technical University of Munich)

Title: A Γ-convergence result for the theory of phase transitions: double-well problem for gradients of vector-valued functions with surfactant

Abstract: We consider energy functionals

$$E_\varepsilon(u,\mu) := \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon|\nabla^2 u|^2 + \varepsilon(\rho - |\nabla^2 u|)^2 dx,$$

where ρ is the density of a measure μ that is absolutely continuos with respect to the Lebesgue measure, u is an vector-valued H^2 function and W is a double-well potential which vanishes for exactly two matrices that are rank-one connected.
In the talk we will motivate the choice of the energy above in the context of the gradient-theory of phase transition and we will state a Γ-convergence result for e_ε as ε vanishes.

Joint work with Marco Cicalese.

Melanie Koser: (Humboldt University of Berlin)

Title: Pattern formation in a two-dimensional frustrated spin system

Abstract: We are interested in pattern formation in two-dimensional magnetic compounds. We consider materials whose atoms are ordered in a regular crystalline structure and associate to each atom its so called spin, a unit vector in $\mathbb{S}^1$. Complex geometric structures in the spin field may be the result of the competition between anti- and ferromagnetic interactions. In ferromagnetic materials spins prefer to be aligned, whereas in antiferromagnetic compounds one cannot observe a global orientation of the spins. The competition between these two interactions leads to frustration mechanisms in the system. We consider the lattice energy of certain materials, in which antiferromagnetic (AF) and ferromagnetic (F) interactions coexist, and are modeled by the J_1-J_3 F-AF model on a square lattice. In this talk we present our current research results. These include a scaling law for the optimal energy, which also describes arising patterns in a minimal spin field. Further, we discuss a Γ-convergence result which in a certain parameter regime relates the discrete model with a suitable continuous counterpart. Next to the vortex free regimes, we also observe regimes which prefer using vortices to lower the interaction energy of a spin field. Further, we present some numerical experiments. (based on a joint work with Janusz Ginster and Barbara Zwicknagl)

Roberta Marziani: (TU Dortmund/University of L'Aquila)

Title: Non-local approximation of free-discontinuity problems in linear elasticity

Abstract: We analyse the Gamma-convergence of general non-local convolution type functionals with varying densities depending on the space variable and on the symmetrized gradient. The limit is a local free-discontinuity functional, where the bulk term can be completely characterized in terms of an asymptotic cell formula. From that, we can deduce an homogenisation result in the stochastic setting. This talk is based on a joint work with Francesco Solombrino.

Fumihiko Onoue: (Technical University of Munich)

Title: Small liquid drops minimizing a long-range interaction energy

Abstract: We consider liquid drops of fixed volume under the action of a potential. At equilibrium, their shape minimizes the free energy which consists of a surface energy plus a potential energy induced by an external force field such as gravity. In this talk, we particularly focus on a surface energy associated with a long-range attractive interaction. We prove that, under small volume constraints, the equilibrium shape of this free energy is "nearly" spherical and convex. This is a joint work with K. Bessas (University of Pavia) and M. Novaga (University of Pisa).

Gianluca Orlando: (Politecnico di Bari)

Title: Chirality transitions in a frustrated spin model

Abstract: Spin models are lattice models that describe magnetic properties of materials. In this talk we will examine a 2-dimensional planar spin model (known as the J1-J2-J3 model) which exhibits frustration. Frustration is the phenomenon due to conflicting interatomic ferromagnetic/antiferromagnetic interactions that prevent the energy of every pair of interacting spins to be simultaneously minimized. The frustration mechanism leads to complex geometric patterns in the material. We study these complex geometric patterns by carrying out a discrete-to-continuum variational analysis as the lattice spacing tends to zero, finding the energetic regime for which many chiral phases can coexist. In particular, we will show that the surface tension between the chiral phases is captured by a continuum energy obtained by suitably selecting solutions to the eikonal equation. The results presented in the seminar are based on works in collaboration with M. Cicalese and M. Forster.

Matthias Ruf: (Ecole polytechnique fédérale de Lausanne)

Title: On the Lavrentiev phenomenon for vector-valued, convex integral functionals

Abstract: The Lavrentiev phenomenon basically says that the infimum value of a functional may depend on the space of admissible functions (e.g., Sobolev, Lipschitz, smooth). For convex autonomous integrands it is believed (and known in the scalar case) that no Lavrentiev phenomenon occurs at least for regular boundary conditions. In this talk we present some new results on the absence of the Lavrentiev phenomenon in the convex, possibly non-autonomous, setting. Joint work with Lukas Koch and Mathias Schäffner.

Giorgio Saracco: (Università di Trento)

Title: Bijections between isoperimetric sets, prescribed curvature sets, and p-Cheeger sets

Abstract: Given a planar, open set Ω, satisfying some weak regular assumptions, we will consider the following three different problems among subsets of Ω: the isoperimetric problem for volume V; the prescribed curvature problem for curvature κ; the p-Cheeger problem.

We shall see that there exist bijections $\mathfrak{K}$ and $\mathfrak{V}$ between these problems in the following sense: a set is isoperimetric of volume V if and only if it attains the minimum of the prescribed curvature functional for curvature $\mathfrak{K}(V)$; analogously a set is p-Cheeger if and only if it is isoperimetric of volume $\mathfrak{V}(p)$.

As a byproduct we infer some convexity properties on the isoperimetric profile, and some fine regularity properties on the contact surface of minimizers.

Based on joint works with Caroccia, Leonardi, Neumayer, and Pratelli.

Antonio Tribuzio: (University of Bonn)

Title: Scaling laws for multi-well nucleation problems

Abstract: In this talk, we study scaling laws for nulceation problems which are motivated by models for shape-memory alloys. More precisely, we discuss optimal energy scaling of inclusions of a phase with several variants (martensite) inside a parent phase (austenite), for various model problems in two and three dimensions. The energy term is given by a singularly-perturbed multi-well elastic energy (without gauge invariances).

We provide scaling results in the volume and the singular perturbation parameter for settings in which the surrounding parent phase is in the first- and the second-order lamination-convex hull of the wells of the "martensite phase". This work is in collaboration with Angkana Rüland.

Konstantinos Zemas: (University of Münster)

Title: Energy barriers for boundary nucleation in a two-well model: The gauge-free case

Abstract: We study scaling laws for a simplified double-well nucleation problem confined in a half-space, without gauge invariances. This is motivated by models for boundary nucleation of a single-phase martensite (described by a rank-one matrix F) inside a parental phase of austenite (described by the zero matrix). Our main focus is how the relationship between the rank-one direction and the orientation of the half-space influences the energy

scaling, with respect to the fixed volume of the inclusion.

This is joint work with Antonio Tribuzio